PARTIAL DIFFERENTIATION

Introduction :-

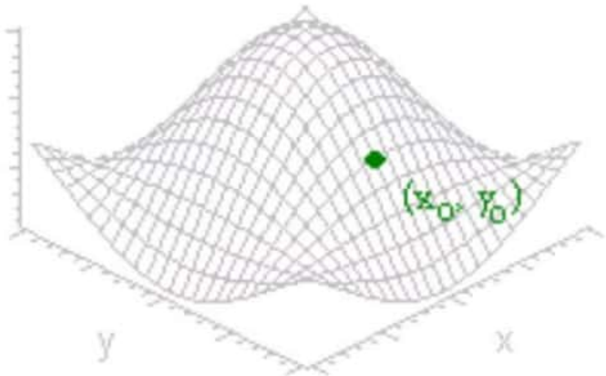
Partial differential equations abound in all branches of science and engineering and many areas of business. The number of applications is endless.

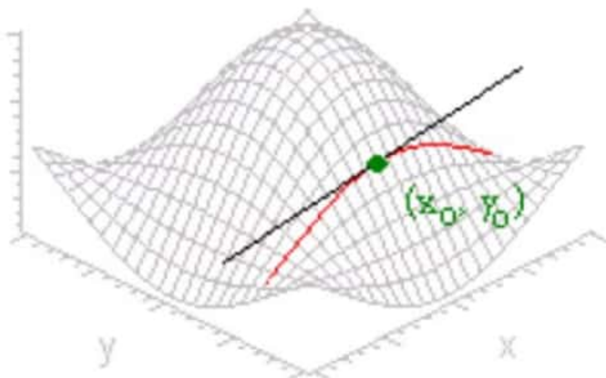
Partial derivatives have many important uses in math and science. We shall see that a partial derivative is not much more or less than a particular sort of directional derivative. The only trick is to have a reliable way of specifying directions ... so most of this note is concerned with formalizing the idea of direction

So far, we had been dealing with functions of a single independent variable. We will now consider functions which depend on more than one independent variable; Such functions are called functions of several variables.

Geometrical Meaning

Suppose the graph of $z = f(x,y)$ is the surface shown. Consider the partial derivative of f with respect to x at a point (x_0, y_0). Holding y constant and varying x, we trace out a curve that is the intersection of the surface with the vertical plane $y = y_0$. The partial derivative $f_x(x_0,y_0)$. measures the change in z per unit increase in x along this curve. That is, $f_x(x_0, y_0)$ is just the slope of the curve at (x_0, y_0). The geometrical interpretation of $f_y(x_0, y_0)$. is analogous.





Real-World Applications:

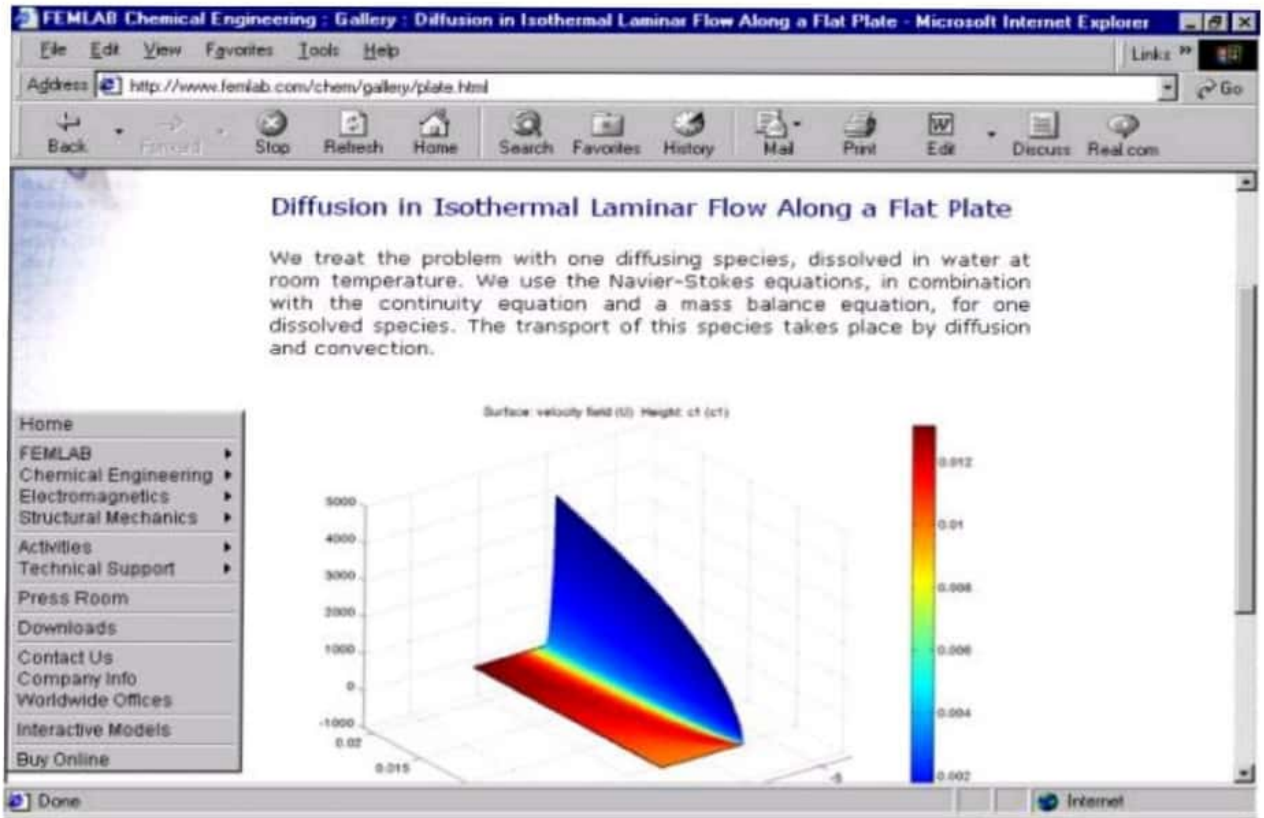
Rates of Change:

In the Java applet we saw how the concept of partial derivative could be applied geometrically to find the slope of the surface in the x and y directions. In the following two examples we present partial derivatives as rates of change. Specifically we explore an application to a temperature function (this example does have a geometric aspect in terms of the physical model itself) and a second application to electrical circuits, where no geometry is involved.

I. Temperature on a Metal Plate

The screen capture below shows a current website illustrating thermal flow for chemical engineering. Our first application will deal with a similar flat plate where temperature varies with position.

* The example following the picture below is taken from the current text in SM221,223: *Multivariable Calculus* by James Stewart.



Suppose we have a flat metal plate where the temperature at a point (x,y) varies according to position. In particular, let the temperature at a point (x,y) be given by,

$$T(x, y) = 60/1 + x^2 + y^2$$

where T is measured in °C and x and y in meters.

Question: what is the rate of change of temperature with respect to distance at the point (2,1) in (a) the x-direction? and (b) in the y-direction ?

Let's take (a) first.

What is the rate of change of temperature with respect to distance at the point (2,1) in (a) the x-direction?

What observations and translations can we make here?

Rate of change of temperature indicates that we will be computing a type of derivative. Since the temperature function is defined on two variables we will be computing a partial derivative. Since the question asks for the rate of change in the x-direction, we will be holding y constant. Thus, our question now becomes:

What is ${dT}/{dx}$ at the point (2,1)?

$$T(x, y) = 60/1 + x^2 + y^2 = 60(1 + x^2 + y^2)^{-1}$$

$$\partial T/\partial x = -60(2x)(1 + x^2 + y^2)^{-2}$$

$$\partial T/\partial x\,(2,1) = -60(4)(1 + 4 + 1)^{-2} = -20/3$$

Conclusion :

The rate of change of temperature in the x-direction at (2,1) is $-20/3$ degrees per meter;

note this means that the temperature is decreasing !

1. If $u = e^{ax-by} \sin(ax+by)$ **show that** $b\frac{\partial u}{\partial x} - a\frac{\partial u}{\partial y} = 2\,ab\,u$

Solution : $u = e^{ax-by} \sin(ax+by)$

$$\therefore \frac{\partial u}{\partial x} = e^{ax-by} \cos(ax+by)\ .\ a + a.e^{ax-by} \sin(ax+by)$$

ie., $\frac{\partial u}{\partial x} = a\, e^{ax-by} \cos(ax+by) + au$ ...(1)

Also $\frac{\partial u}{\partial x} = e^{ax-by} \cos(ax+by)\ .\, b + (-b)e^{ax-by} \sin(ax+by)$

ie., $\frac{\partial u}{\partial y} = b\, e^{ax-by} \cos(ax+by) - bu$ ...(2)

Now $b\frac{\partial u}{\partial x} = a\frac{\partial u}{\partial y}$ by using (1) and (2) becomes

$$= abe^{ax-by} \cos(ax+by) + abu - abe^{ax-by} \cos(ax+by) + abu$$

$= 2$ abu

Thus $b\frac{\partial u}{\partial x} - a\frac{\partial u}{\partial y} = 2\,abu$

2. If $u = e^{ax+by} f(ax-by)$, prove that

$$b\frac{\partial u}{\partial x} = a\frac{\partial u}{\partial y} = 2abu$$

Solution : $u = e^{ax+by} f(ax-by)$, by data

$$\frac{\partial u}{\partial x} = e^{ax+by}\,.f'(ax-by)a + ae^{ax+by}f(ax-by)$$

Or $\frac{\partial u}{\partial x} = a\, e^{ax+by}\,.f'(ax-by) + a\,u$ ...(1)

Next, $\frac{\partial u}{\partial y} = e^{ax+by} f'(ax-by).\,(-b) + b\, e^{ax+by} f(ax-by)$

Or $\frac{\partial u}{\partial y} = -b\, e^{ax+by} f'(ax-by) + ba$ ...(2)

Now consider L.H.S $= b\frac{\partial u}{\partial x} + a\frac{\partial u}{\partial y}$

$= b\left\{ae^{ax+by} f'(ax - by) + au\right\} + a\left\{- be^{ax+by} f'(ax - by) + bu\right\}$

$= ab\, e^{ax+by} f'(ax - by) + abu - ab\, e^{ax+by} f'(ax - by) + abu$

= 2abu = R.H.S

Thus $b\frac{\partial u}{\partial x} + a\frac{\partial u}{\partial y} = 2abu$

3. **If** $u = \log\sqrt{x^2 + y^2 + z^2}$ **, show that** $(x^2 + y^2 + z^2)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 1$

Solution : By data $u = \log\sqrt{x^2 + y^2 + z^2} = \frac{1}{2}\log(x^2 + y^2 + z^2)$

The given u is a symmetric function of x, y, z,

(It is enough if we compute only one of the required partial derivative)

$$\frac{\partial u}{\partial x} = \frac{1}{2} \cdot \frac{1}{x^2 + y^2 + z^2} \cdot 2x = \frac{x}{x^2 + y^2 + z^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2 + z^2}\right)$$

ie., $\frac{(x^2 + y^2 + z^2)1 - x \cdot 2x}{(x^2 + y^2 + z^2)^2} = \frac{y^2 + z^2 - x^2}{(x^2 + y^2 + z^2)^2}$

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{y^2 + z^2 - x^2}{(x^2 + y^2 + z^2)^2} \qquad \text{...(1)}$$

Similarly $\therefore \frac{\partial^2 u}{\partial y^2} = \frac{z^2 + x^2 - y^2}{(x^2 + y^2 + z^2)^2}$ ...(2)

$$\frac{\partial^2 u}{\partial z^2} = \frac{x^2 + y^2 - z^2}{(x^2 + y^2 + z^2)^2} \qquad \text{...(3)}$$

Adding (1), (2) and (3) we get,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^2} = \frac{1}{x^2 + y^2 + z^2}$$

Thus $(x^2 + y^2 + z^2)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 1$

4. **If u = log (tan x + tan y + tan z), show that,**

$\sin 2x\, u_x + \sin 2y\, u_y + \sin 2z\, u_z = 2$

Solution : u = log (tan x + tan y + tan z) is a symmetric function.

$$u_x = \frac{\sec^2 x}{\tan x + \tan y + \tan z}$$

$$\sin 2x\, u_x = \frac{(2\sin x \cos x)\sec^2 . x}{\tan x + \tan y + \tan z}$$

Or $$\sin 2x\, u_x = \frac{2\tan x}{\tan x + \tan y + \tan z} \qquad ...(1)$$

Similarly $$\sin 2y\, u_y = \frac{2\tan y}{\tan x + \tan y + \tan z} \qquad ...(2)$$

$$\sin 2z\, u_y = \frac{2\tan z}{\tan x + \tan y + \tan z} \qquad ...(3)$$

Adding (1), (2) and (3) we get,

$$\sin 2x\, u_x + \sin 2y\, u_y + \sin 2z\, u_z = \frac{2(\tan x + \tan y + \tan z)}{(\tan x + \tan y + \tan z)} = 2$$

Thus $\sin 2x\, u_x + \sin 2y\, u_y + \sin 2z\, u_z = 2$

5. **If u = log ($x^3 + y^3 + z^3 - 3xyz$) then prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$ and hence show that** $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$

Solution : u = log ($x^3 + y^3 + z^3 - 3xyz$) is a symmetric function

$$\frac{\partial u}{\partial x} = \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz} \qquad ...(1)$$

$$\frac{\partial u}{\partial y} = \frac{3y^2 - 3zx}{x^3 + y^3 + z^3 - 3xyz} \qquad ...(2)$$

$$\frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz} \qquad ...(3)$$

Adding (1), (2) and (3) we get,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3\left(x^2 + y^2 + z^2 - xy - yz - zx\right)}{\left(x^3 + y^3 + z^3 - 3xyz\right)}$$

Recalling a standard elementary result,

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)\left(a^2 + b^2 + c^2 - ab - bc - ca\right)$$

We have,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3\left(x^2 + y^2 + z^2 - xy - yz - zx\right)}{(x + y + z)\left(x^2 + y^2 + z^2 - xy - yz - zx\right)}$$

Thus $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x + y + z}$

Further $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) u$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x + y + z}\right)$$, by using the earlier result.

$$= \frac{\partial}{\partial x}\left(\frac{3}{x + y + z}\right) + \frac{\partial}{\partial y}\left(\frac{3}{x + y + z}\right) + \frac{\partial}{\partial z}\left(\frac{3}{x + y + z}\right)$$

$$= \frac{-3}{(x + y + z)^2} + \frac{-3}{(x + y + z)^2} + \frac{-3}{(x + y + z)^2} = \frac{-9}{(x + y + z)^2}$$

Thus $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x + y + z)^2}$

6. If $u = f(r)$ and $x = r\cos\theta$, $y = r\sin\theta$,

prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = f''(r) + \dfrac{1}{r} f'(r)$

Solution : Observing the required partial derivative we conclude that u must be a function of x, y. But u = f(r) by data and hence we need to have r as a function of x, y. Since $x = r\cos\theta$, $y = r\sin\theta$ we have $x^2 + y^2 = r^2$.

$\therefore$ we have $u = f(r)$ where $r = \sqrt{x^2 + y^2}$

$$\frac{\partial^2 u}{\partial x^2} = \frac{f'(r)}{r^3}\left(r^2 - x^2\right) + \frac{f''(r)}{r^2} \cdot x^2 \text{ and}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{f'(r)}{r^3}\left(r^2 - y^2\right) + \frac{f''(r)}{r^2} \cdot y^2$$

Adding these results we get,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{f'(r)}{r^3}\left\{2x^2 - \left(x^2 + y^2\right)\right\} + \frac{f''(r)}{r^2}\left(x^2 + y^2\right)$$

$$= \frac{f'(r)}{r^3} \cdot r^2 + \frac{f''(r)}{r^2} \cdot r^2 = \frac{1}{r} f'(r) + f''(r)$$

Thus $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = f''(r) + \dfrac{1}{r} f'(r)$

7. Prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = nu$

Proof : Since u = f (x, y) is a homogeneous function of degree n we have by the definition,

$$u = x^n g(y/x) \qquad \text{...(1)}$$

Let us differentiate this w.r.t x and also w.r.t.y

$$\therefore \frac{\partial u}{\partial x} = x^n \cdot g'(y/x) \cdot \left(-\frac{y}{x^2}\right) + nx^{n-1} g(y/x)$$

ie., $$\frac{\partial u}{\partial x} = x^{n-2}\, y\, g'(y/x) + nx^{n-1} g(y/x) \qquad \text{...(2)}$$

Also $$\frac{\partial u}{\partial y} = x^n \cdot g'(y/x) \cdot \left(\frac{1}{x}\right)$$

ie., $\frac{\partial u}{\partial y} = x^{n-1} . g'(y/x)$ ...(3)

Now consider $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}$ as a consequence of (2) and (3)

$= x\left[- x^{n-2} y g'(y/x) + n x^{n-1} g(y/x)\right] + y\left[x^{n-1} g'(y/x)\right]$

$= -x^{n-1} y g'(y/x) + n x^n g(y/x) + x^{n-1} y g'(y/x)$

$= n . x^n g(y/x)$

= n u, by using (1)

Thus we have proved Euler's theorem

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n u ; x u_x + y u_y = n u$$

8. **Prove that** $x^2 \frac{\partial^2 u}{\partial x^2} + 2 x y \frac{\partial^2 u}{\partial x \partial y} = n(n-1) u$

Proof : Since u = f (x, y) is a homogeneous function of degree n, we have Euler's theorem

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n u \quad ...(1)$$

Differentiating (1) partially w.r.t. x and also w.r.t y we get,

$$\left(x \frac{\partial^2 u}{\partial x^2} + 1. \frac{\partial u}{\partial x}\right) + y \frac{\partial^2 u}{\partial x \partial y} = n \frac{\partial u}{\partial x} \quad ...(2)$$

Also, $x \frac{\partial^2 u}{\partial y \partial x} + \left(y \frac{\partial^2 u}{\partial y^2} + 1 . \frac{\partial u}{\partial y}\right) = n \frac{\partial u}{\partial y}$ ...(3)

We shall now multiply (2) by x and (3) by y.

$$x^2 \frac{\partial^2 u}{\partial x^2} + x \frac{\partial u}{\partial x} + x y \frac{\partial^2 u}{\partial x \partial y} = n x \frac{\partial u}{\partial x} \text{ and}$$

$$x y \frac{\partial^2 u}{\partial y \partial x} + y^2 \frac{\partial^2 u}{\partial y^2} + y \frac{\partial u}{\partial y} = ny \frac{\partial u}{\partial y}$$

Adding these using the fact that $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial y \partial x}$ we get,

$$\left(x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}\right)+\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=n\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

ie., $x^2\frac{\partial^2 u}{\partial x^2}+2\,x\,y\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+n\,u=n\,(n\,u)$, by using (1)

or $x^2\frac{\partial^2 u}{\partial x^2}+2\,x\,y\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+n\,(nu)-nu=n\,(n-1)\,u$

Thus $x^2\frac{\partial^2 u}{\partial x^2}+2\,x\,y\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+\;n\,(n-1)\,u$

ie., $x^2u_{xx}+2\,x\,y\,u_{xy}+y^2u_{yy}=n\,(n-1)\,u$

9. **If** $u=\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}$ **show that** $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0$

Solution : (Observe that the degree is 0 in every term)

$$u=\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}$$

We shall divide both numerator and denominator of every term by x.

$$u=\frac{1}{y/x+z/x}+\frac{y/x}{z/x+1}+\frac{z}{1+y/x}=x^{\circ}\{g\,(y/x,\,z/x)\}$$

$\Rightarrow$ u is homogeneous of degree 0. $\therefore$ n = 0

We have Euler's theorem, $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=n\,u$

Putting n = 0 we get, $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0$

10. **If** $u=\log\left(\frac{x^4+y^4}{x+y}\right)$ show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3$

Solution : we cannot put the given u in the form x^n g (y/x)

$$\therefore\ e^u=\frac{x^4+y^4}{x+y}=\frac{x^4\left(1+y^4/x^4\right)}{x\left(1+y/x\right)}=x^3\left\{\frac{1+(y/x)^4}{1+(y/x)}\right\}$$

ie., $e^u = x^3$ g (y/x) $\Rightarrow$ e^u is homogeneous of degree 3 $\therefore$ n = 3

Now applying Euler's theorem for the homogeneous function e^u

We have $x\dfrac{\partial(e^u)}{\partial x} + y\dfrac{\partial(e^u)}{\partial y} = n\,e^u$

ie., $x\,e^u\dfrac{\partial u}{\partial x} + y\,e^u\dfrac{\partial u}{\partial y} = 3e^u$

Dividing by e^u we get $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3$

11. If $u = \tan^{-1}\left(\dfrac{x^3+y^3}{x-y}\right)$ **show that**

(i) $x\,u_x + y\,u_y = \sin 2u$

(ii) $x^2u_{xx} + 2xy\,u_{xy} + y^2u_{yy} = \sin 4u - \sin 2u$

Solution : (i) $u = \tan^{-1}\left(\dfrac{x^3+y^3}{x-y}\right)$ by data

$$\Rightarrow \tan u = \frac{x^3+y^3}{x-y} = \frac{x^3(1+y^3/x^3)}{x(1-y/x)} = x^2\left\{\frac{1+(y/x)^3}{1-(y/x)}\right\}$$

ie., tan u = x^2/g (y/x) $\Rightarrow$ tan u is homogeneous of degree 2.

Applying Euler's theorem for the function tan u we have,

$$x\frac{\partial(\tan u)}{\partial x} + y\frac{\partial(\tan u)}{\partial y} = n\,.\tan u\,;\, n=2$$

ie., $x\sec^2 u\dfrac{\partial u}{\partial x} + y\sec^2 u\dfrac{\partial u}{\partial y} = 2\tan u$

or $\quad x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{2\tan u}{\sec^2 u} = 2\cos^2 u\dfrac{\sin u}{\cos u} = 2\cos u\sin u = \sin 2u$

$\therefore\; xu_x + yu_y = \sin 2u$

(ii) We have $xu_x + y\,u_y = \sin 2u$ ...(1)

Differentiating (1) w.r.t x and also w.r.t y partially we get

$x\,u_{xx} + 1\,.\,u_x + yu_{xy} = 2\cos 2u\,.\,u_x$ ...(2)

And $x\,u_{yx} + yu_{yy} + 1\,.\,u_y = 2\cos 2u\,.\,u_y$ ...(3)

Multiplying (2) by x and (3) by y we get,

$x^2u_{xx} + x\,u_x + xy\,u_{xy} = 2\cos 2u.\,xu_x$

$$xy\,u_{yx} + y^2 u_{yy} + y\,u_y = 2\cos 2u.\ yu_y$$

Adding these by using the fact that $u_{yx} = u_{xy}$, we get

$$x^2 u_{xx} + 2xy\,u_{xy} + y^2 u_{yy} + (xy_x + yu_y) = 2\cos 2u\,(xu_x + y\,u_y)$$

By using (1) we have,

$$x^2 u_{xx} + 2xy\,u_{xy} + y^2 u_{yy} = 2\cos 2u\ \sin 2u - \sin 2u$$

(since $\sin 2\theta = 2\cos\theta\sin\theta$, first term in the R.H.S becomes $\sin 4u$)

Thus $x^2u_{xx} + 2xy\,u_{xy} + y^2 u_{yy} = \sin 4u - \sin 2u$

12. If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$ **Prove that** $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$

>> here we need to convert the given function u into a composite function.

Let $u = f(p, q, r)$ where $p = \frac{x}{y}, q = \frac{y}{z}, r = \frac{z}{x}$

ie., $\{u \to (p, q, r) \to (x, y, z)\} \Rightarrow u \to x, y, z$

$$\therefore\ \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial x}$$

ie., $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\ .\ \frac{1}{y} + \frac{\partial u}{\partial q}\ .\ 0 + \frac{\partial u}{\partial r}\ .\left(-\frac{z}{x^2}\right)$

$$\therefore\ x\frac{\partial u}{\partial x} = \frac{x}{y}\frac{\partial u}{\partial p} - \frac{z}{x}\frac{\partial u}{\partial r} \qquad ...(1)$$

Similarly by symmetry we can write,

$$y\frac{\partial u}{\partial y} = \frac{y}{z}\frac{\partial u}{\partial q} - \frac{x}{y}\frac{\partial u}{\partial p} \qquad ...(2)$$

$$z\frac{\partial u}{\partial z} = \frac{z}{x}\frac{\partial u}{\partial r} - \frac{y}{z}\frac{\partial u}{\partial q} \qquad ...(3)$$

Adding (1), (2) and (3) we get $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$

13. **If u = f(x – y, y – z, z – x) show that** $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$

>> Let u = f(p, q, r) where p = x – y, q = y – z, r = z – x

$$\therefore \frac{\partial u}{\partial x}=\frac{\partial u}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial u}{\partial q}\frac{\partial q}{\partial x}+\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}$$

ie., $\frac{\partial u}{\partial x}=\frac{\partial u}{\partial p}\cdot 1+\frac{\partial u}{\partial q}\cdot 0+\frac{\partial u}{\partial r}(-1)$

$$\therefore \frac{\partial u}{\partial x}=\frac{\partial u}{\partial p}-\frac{\partial u}{\partial r} \qquad ...(1)$$

Similarly we have by symmetry

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial q}-\frac{\partial u}{\partial p} \qquad ...(2)$$

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial r}-\frac{\partial u}{\partial q} \qquad ...(3)$$

Adding (1), (2) and (3) we get, $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$

14. **If z = f(x, y) where x = r cos θ and y = r sin θ**

Show that $\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$

Solution : $\{z \to (x, y) \to (r, \theta)\} \Rightarrow z \to (r, \theta)$

$$\therefore \frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial r};\frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}$$

ie., $\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}\cos\theta+\frac{\partial z}{\partial y}\sin\theta \qquad ...(1)$

and $\frac{\partial z}{\partial \theta}=\frac{\partial z}{\partial x}(-r\sin\theta)+\frac{\partial z}{\partial y}(r\cos\theta)=r\left[\frac{-\partial z}{\partial x}\sin\theta+\frac{\partial z}{\partial y}\cos\theta\right]$

or $\frac{1}{r}=\frac{\partial z}{\partial \theta}=\frac{-\partial z}{\partial x}\sin\theta+\frac{\partial z}{\partial y}\cos\theta$

squaring and adding (1), (2) and collecting suitable terms have,

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 \left[\cos^2\theta + \sin^2\theta\right]$$

$$+ \left(\frac{\partial z}{\partial y}\right)^2 \left[\sin^2\theta + \cos^2\theta\right] + 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\cos\theta\sin\theta - 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\sin\theta\cos\theta$$

$$\therefore \left(\frac{\partial z}{\partial r}\right)^2 \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 \quad \text{ie., R.H.S = L.H.S}$$

15. **If** $z = f(x, y)$ where $x = e^u + e^{-v}$, $y = e^{-u} - e^v$

Prove that $x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}$

Solution : $\{z \to (x, y) \to (u, v)\} \Rightarrow z \to (u, v)$

$$\therefore \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}; \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$$ie., \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot e^u + \frac{\partial z}{\partial y}\left(-e^{-u}\right) \qquad ...(1)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\left(-e^{-v}\right) + \frac{\partial z}{\partial y}\left(-e^{v}\right) \qquad ...(2)$$

Consider R.H.S = $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}$ and (1) – (2) yields

$$\frac{\partial z}{\partial x}\left(e^u + e^{-v}\right) - \frac{\partial z}{\partial y}\left(e^{-u} - e^v\right) = \frac{\partial z}{\partial x} \cdot x - \frac{\partial z}{\partial y} \cdot y$$

Thus $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial y} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$ ie., R.H.S = L.H.S

16. **Find** $\frac{\partial(u, v, w)}{\partial(x, y, z)}$ **where** $u = x^2 + y^2 + z^2$, $v = xy+yz+zx$, $w=x+y+z$

Solution : The definition of $J = \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$

But $u = x^2 + y^2 + z^2$, $v = xy+yz+zx$, $w=x+y+z$

Substituting for the partial derivatives we get

$$J = \begin{vmatrix} 2x & 2y & 2z \\ y+z & x+z & y+x \\ 1 & 1 & 1 \end{vmatrix}$$

Expanding by the first row,

J = 2x {(x + z) – (y + x)} -2y {(y + z) – (y + x)}

+2z {(y + z) – (x + z)}

= 2x (z-y) – 2y(z-x) + 2z(y-x)

= 2(xz – xy – yz + xy + yz – xz) = 0 Thus **J = 0**

17. **If** $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$, **show that** $\frac{\partial(u, v, w)}{\partial(x, y, z)} = 4$

Solution : by data $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{-yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & \frac{-zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & \frac{-xy}{z^2} \end{vmatrix}$$

$$= \frac{-yz}{x^2}\left\{\left(\frac{-zx}{y^2}\right)\left(\frac{-xy}{z^2}\right) - \left(\frac{x}{z}\right)\left(\frac{x}{y}\right)\right\}$$

$$- \frac{z}{x}\left\{\frac{z}{y}\left(\frac{-zx}{y^2}\right) - \frac{y}{z}\cdot\frac{x}{y}\right\} + \frac{y}{x}\left\{\frac{z}{y}\cdot\frac{x}{z} - \frac{y}{z}\left(\frac{-zx}{y^2}\right)\right\}$$

$$= \frac{-yz}{x^2}\left\{\frac{x^2}{yz} - \frac{x^2}{yz}\right\} - \frac{z}{x}\left\{\frac{-x}{z} - \frac{x}{z}\right\} + \frac{y}{x}\left\{\frac{x}{y} + \frac{x}{y}\right\}$$

= 0+1+1+1+1=4

Thus $\frac{\partial(u, v, w)}{\partial(x, y, z)} = 4$

18. **If u + v = e^x cos y and u – v = e^x sin y find the jacobian of the functions u and v w.r.t x and y.**

Solution : we have to find $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$

Using the given data we have to solve for u and v in terms of x and y.

By data $u + v = e^x \cos y$ ......(1)

$u - v = e^x \sin y$ ......(2)

(1) + (2) gives : $2u = e^x(\cos y + \sin y)$

(2) – (2) gives : $2v = e^x(\cos y - \sin y)$

Ie., $u = \dfrac{e^x}{2}(\cos y + \sin y)$; $v = \dfrac{e^x}{2}(\cos y - \sin y)$

$\therefore \dfrac{\partial u}{\partial x} = \dfrac{e^x}{2}(\cos y + \sin y)$, $\dfrac{\partial v}{\partial x} = \dfrac{e^x}{2}(- \sin y - \cos y)$

Now $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{e^x}{2}(\cos y + \sin y) & \dfrac{e^x}{2}(-\sin y + \cos y) \\ \dfrac{e^x}{2}(\cos y - \sin y) & \dfrac{-e^x}{2}(\sin y + \cos y) \end{vmatrix}$

$= \dfrac{e^x}{2} \cdot \dfrac{e^x}{2} \{ -(\cos y + \sin y)^2 - (\cos y - \sin y)^2 \}$

$= \dfrac{-e^{2x}}{4} \{1 + \sin 2y) + (1 - \sin 2y)\} = \dfrac{-e^{2x}}{2}$

Thus $\dfrac{\partial(u,v)}{\partial(x,y)} = \dfrac{-e^{2x}}{2}$

19. (a) If x = r cos θ, y = r sin θ find the value of $\frac{\partial(r,\theta)}{\partial(x,y)}$

(b) Further verify that $\frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = 1$

(a) **Solution :** We shall first express r, θ in terms of x and y.

We have x = r cos θ, y = r sin θ by data.

$\therefore x^2 + y^2 = r^2$ and $\frac{y}{x} = \tan\theta$ or $\theta = \tan^{-1}(y/x)$

Consider $r^2 = x^2 + y^2$

Differentiating partially w.r.t x and also w.r.t y we get,

$$2r\frac{\partial r}{\partial x} = 2x \quad \text{and} \quad 2r\frac{\partial r}{\partial y} = 2y$$

$$\therefore \quad \frac{\partial r}{\partial x} = \frac{x}{r} \quad \text{and} \quad \frac{\partial r}{\partial y} = \frac{y}{r}$$

Also consider $\theta = \tan^{-1}(y/x)$

$$\therefore \quad \frac{\partial\theta}{\partial x} = \frac{1}{1+(y/x)^2} \cdot \frac{-y}{x^2} \quad \text{and} \quad \frac{\partial\theta}{\partial y} = \frac{1}{1+(y/x)^2} \cdot \frac{1}{x}$$

$$\text{i.e.,} \quad \frac{\partial\theta}{\partial x} = \frac{-y}{x^2+y^2} \quad \text{and} \quad \frac{\partial\theta}{\partial y} = \frac{x}{x^2+y^2}$$

$$\text{Now } \frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial r} & \frac{\partial r}{dy} \\ \frac{\partial\theta}{dx} & \frac{\partial\theta}{dy} \end{vmatrix} = \begin{vmatrix} \frac{x}{r} & \frac{y}{r} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix}$$

$$i.e., \quad = \frac{x^2}{r(x^2+y^2)} + \frac{y^2}{r(x^2+y^2)} = \frac{(x^2+y^2)}{r(x^2+y^2)} = \frac{1}{r}$$

$$\therefore \frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{r}$$

Solution (b) : Consider $x = r\cos\theta, y = r\sin\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{d\theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r$$

$$\therefore \frac{\partial(x,y)}{\partial(r,\theta)} = r$$

From (1) and (2) : $\frac{\partial(x,y)}{\partial(x,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = r \cdot \frac{1}{r} = 1$

20. **If** $x = u(1-v), y = uv$ **then show that** $JJ' = 1$

Solution : $x = u(1-v);\ y = uv$

$$\frac{\partial x}{\partial u} = (1-v),\ \frac{\partial y}{\partial u} = v \qquad \frac{\partial x}{\partial v} = -u, \frac{\partial y}{\partial v} = u$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} (1-v) & -u \\ v & u \end{vmatrix}$$

$$= (1-v)u + uv = u \quad \therefore \quad J = u$$

Next we shall express u and v in terms of x and y.

By data x = u - uv and y = uv

Hence $x + y = u$. Also $v = \frac{y}{u} = \frac{y}{x+y}$

Now we have, $u = x + y; v = \frac{y}{x+y}$ $\qquad \therefore \quad \frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial y} = 1,$

$$\frac{\partial v}{\partial x} = \frac{(x+y)\cdot 0 - y\cdot 1}{(x+y)2} = \frac{x}{(x+y)^2}$$

$$\therefore J' = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ \frac{-y}{(x+y)^2} & \frac{x}{(x+y)^2} \end{vmatrix}$$

$$= \frac{x}{(x+y)^2} + \frac{y}{(x+y)^2} = \frac{x+y}{(x+y)^2} = \frac{1}{(x+y)} = \frac{1}{u}$$

Thus $J' = \frac{1}{u}$ Hence $J \cdot J' = u \cdot \frac{1}{u}$ Thus $JJ' = 1$

21. State Taylor's Theorem for Functions of Two Variables.

Statement: Considering f (x + h, y + k) as a function of a single variable x, we have by Taylor's Theorem

$$f(x+h, y+k) = f(x, y+k) + h\,\frac{\partial f(x,y+k)}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 f(x,y+k)}{\partial x^2} + \text{---------}(1)$$

Now expanding f (x, y + k) as function of y only,

$$f(x, y+k) = f(x,y) + k\,\frac{\partial f(x,y)}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 f(x,y)}{\partial y^2} + \text{--------}$$

$\therefore$ (i) takes the form $f(x+h, y+k) = f(x,y) + k\,\frac{\partial f(x,y)}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 f(x,y)}{\partial y^2} + \text{----------------------------------} +$

$$h\,\frac{\partial}{\partial x}\left\{ f(x,y) + k\frac{\partial f(x,y)}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 f(x,y)}{\partial y^2} + -----\right\}$$

$$+ \frac{h^2}{2!}\frac{\partial^2}{\partial x^2}\left\{ f(x,y) + k\frac{\partial(x,y)}{\partial y} + --------\right\}$$

Hence $f(x+h, y+k) = f(x, y) + h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right) + --\ (1)$

In symbol we write it as

$$F(x+h, y+k) = f(x,y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f + ----$$

Taking x = a and y = b, (1) becomes

$$f(a+h, b+k) = f(a,b) + [h f_x(a,b) + k f_y(a,b)] + \frac{1}{2!}[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b)$$

$$+ k^2 f_{yy}(a,b)] + ---------$$

Putting a + h = x and b + k = y so that h = x – a, k = y – b, we get

$$F(x,y) = f(a,b) + [(x-a) f_x(a,b) + (y-b) f_y(a,b)]$$

$$= \frac{1}{2!}[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)] + ------ (2)$$

This is Taylor's expansion of f (x,y) in powers of (x – a) and (y – b). It is used to expand f (x,y) in the neighborhood of (a,b)

corollary, putting a = 0, b = 0 in (2), we get

$$f(x,y) = f(0,0) + [x f_x(0,0) + y f_y(0,0)] + \frac{1}{2!}[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0)$$

$$+ y^2 f_{yy}(0,0)] + --------- (3)$$

This is Maclaurin's Expansion of f (x,y)

22. Expand $e^x \log(1 + y)$ in powers of x and y up to terms of third degree.

Solution: Here

$f(x,y) = e^x \log(1+y) \quad \therefore f(0,0) = 0$

$f_x(x,y) = e^x \log(1+y) \quad \therefore f_x(0,0) = 0$

$f_y(x,y) = e^x \dfrac{1}{1+y} \quad \therefore f_y(0,0) = 1$

$f_{xx}(x,y) = e^x \log(1+y) \quad \therefore f_{xx}(0,0) = 0$

$f_{xy}(x,y) = e^x \dfrac{1}{1+y} \quad \therefore f_{xy}(0,0) = 1$

$f_{yy}(x,y) = -e^x (1+y)^{-2} \quad \therefore f_{yy}(0,0) = -1$

$f_{xxx}(x,y) = e^x \log(1+y) \quad \therefore f_{xxx}(0,0) = 0$

$f_{xxy}(x,y) = e^x \dfrac{1}{1+y} \quad \therefore f_{xxy}(0,0) = 1$

$f_{xyy}(x,y) = -e^x (1+y)^2 \quad \therefore f_{xyy}(0,0) = -1$

$f_{yyy}(x,y) = 2e^x (1+y)^{-3} \quad \therefore f_{yyy}(0,0) = 2$

Now, Maclaurin's expansion of f (x,y) gives

$$f(x,y) = f(0,0) + x(f x(0,0) + y f_y(0,0) + \frac{1}{2!}\{x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) +$$

$$y^2 f_{yy}(0,0)\} + \frac{1}{2!}\{x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\} + \text{-----------}$$

$$\therefore e^x \log(1+y) = 0 + x.0 + y(1) + \frac{1}{2!}\{x2.0 + 2xy(1) + y2(-1)\}$$

$$+ \frac{1}{2!}\{x^3.0 + 3x^2y(1) + 3xy^2(-1) + y^3(2)\} + \text{---------}$$

$$= y + xy - \frac{1}{2}y^2 + \frac{1}{2}(x^2y = xy^2) + \frac{1}{2}y^3 + \text{----------}$$

23. Expand f (x,y) = e^x Cosy by Taylor's Theorem about the point $\left(1, \frac{\pi}{4}\right)$ up to the Second degree terms.

Solution: f (x,y) = e^x Cosy and a = 1, b = $\frac{\pi}{4}$ $\quad \therefore f = \left(1, \frac{\pi}{4}\right) = e/\sqrt{2}$

$f_x(x,y) = e^x \text{Cos } y \quad \therefore f\left(1, \frac{\pi}{4}\right) = \frac{e}{\sqrt{2}}$

$f_y(x,y) = -e^x \text{Sin } y \quad \therefore f_y\left(1, \frac{\pi}{4}\right) = -\frac{e}{\sqrt{2}}$

$f_{xx}(x,y) = e^x \text{Cos } y \quad \therefore f_{xx}\left(1, \frac{\pi}{4}\right) = \frac{e}{\sqrt{2}}$

$f_{xy}(x,y) = -e^x \text{Sin } y \quad \therefore f_{xy}\left(1, \frac{\pi}{4}\right) = -\frac{e}{\sqrt{2}}$

$f_{yy}(x,y) = -e^x \text{Cos } y \quad \therefore f_{yy}\left(1, \frac{\pi}{4}\right) = -\frac{e}{\sqrt{2}}$

Hence by Taylor's Theorem, we obtain

$$f(x,y) = f\left(1, \frac{\pi}{4}\right) + \left[(x-1) f_x + \left(y - \frac{\pi}{4}\right) fy\right] +$$

$$\frac{1}{2!}\left[(x-1)^2 f_x + 2(x-1)\left(y - \frac{\pi}{4}\right)^2 f_{yy}\right] + \text{-------------}$$

$$\text{i.e., } e^x \text{Cosy} = \frac{e}{\sqrt{2}} + \left[(x-1)\frac{e}{\sqrt{2}} + \left(y - \frac{\pi}{4}\right)\left(-\frac{e}{\sqrt{2}}\right)\right] + \frac{1}{2!}$$

$$\left[(x-1)^2 \frac{e}{\sqrt{2}} + 2(x-1)\left(y - \frac{\pi}{4}\right)\left(-\frac{e}{\sqrt{2}}\right) + \left(y - \frac{\pi}{4}\right)^2\left(\frac{-e}{\sqrt{2}}\right)\right] + \text{------------}$$

$$e^x \text{Cosy} = \frac{e}{\sqrt{2}}\left[1 + (x-1) - \left(y - \frac{\pi}{4}\right)\right] + \frac{1}{2!}\left[(x-1)^2 - 2(x-1)\left(y - \frac{\pi}{4}\right) - \left(y - \frac{\pi}{4}\right)^2\right] + \text{-------}\}$$

Exercise:

1) Expand e^{xy} up to Second degree terms by using Maclaurin's theorem

2) Expand Log (1 – x – y) up to Third degree terms by using Maclaurin's theorem

3) Expand x^2y about the point (1,-2) by Taylor's expansion

4) Obtain the Taylor's expansion of e^x Siny about the point $\left(0, \pi/2\right)$ up to Second degree terms

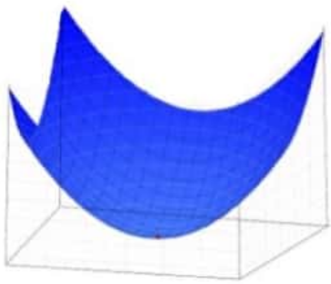
5) Expand $e^{\sin x}$ up to the term containing x4

Maxima and Minima:-

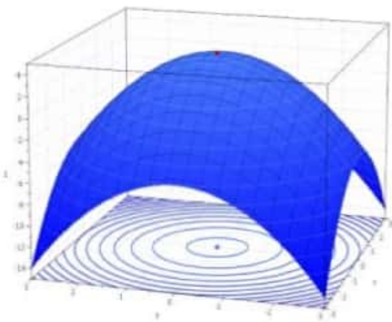
In mathematics, the maximum and minimum (plural: maxima and minima) of a function, known collectively as extrema (singular: extremum), are the largest and smallest value that the function takes at a point within a given neighborhood.

A function *f* (*x*, *y*) is said to have a Maximum value at (*a*,*b*) if their exists a neighborhood point of (*a*,*b*) (say (*a*+*h*, *b*+*k*)) such that *f* (*a*, *b*) > *f* (*a*+*h*, *b*+*k*).
Similarly,
Minimum value at (*a*,*b*) if there exists a neighborhood point of (*a*,*b*) (say (*a*+*h*, *b*+*k*)) such that *f* (*a*, *b*) < *f* (*a*+*h*, *b*+*k*).

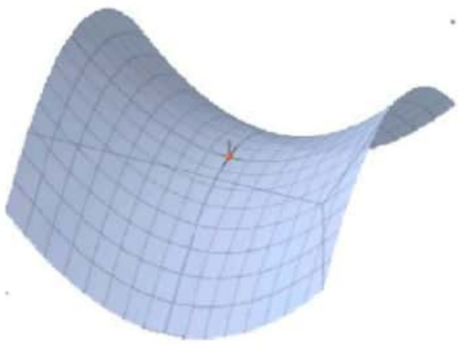
A Minimum point on the graph (in red) $f(x, y) = x^2 + y^2(1 - x)^3$



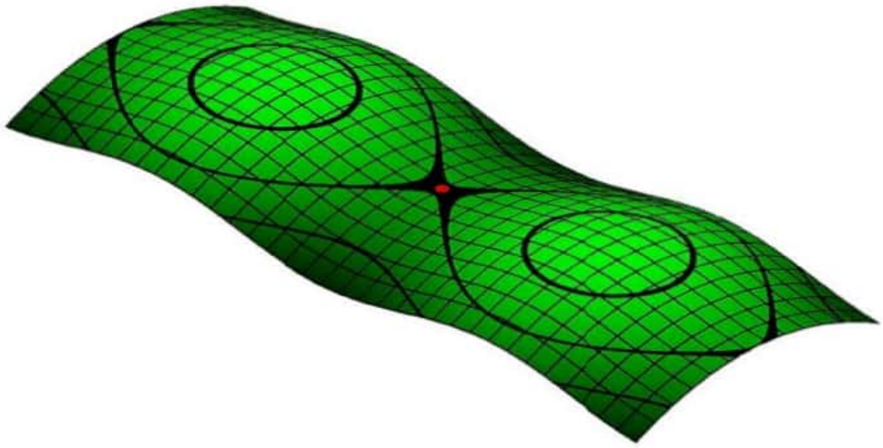
A Maximum point on the graph is at the top (in red)



A saddle point on the graph of $z=x^2-y^2$ (in red)



Saddle point between two hills.



Necessary and Sufficient Condition:-

- If $f_x = 0$ and $f_y = 0$ (Necessary Condition)
- Function will be minimum if $AC\text{-}B^2 > 0$ *and* $A > 0$
- Function will be maximum if $AC\text{-}B^2 > 0$ *and* $A < 0$
- Function will be neither maxima nor minima if $AC\text{-}B^2 < 0$
- If $AC\text{-}B^2 = 0$ we cannot make any conclusion without any further analysis

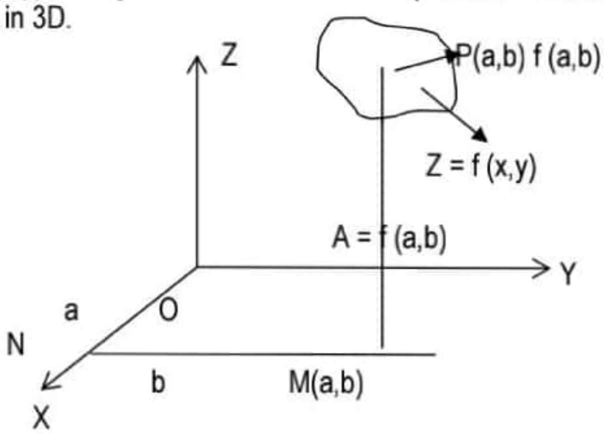
where $A = f_{xx}$, $B = f_{xy}$, $C = f_{yy}$

Working Procedure:-

- First we find Stationary points by considering $f_x = 0$ and $f_y = 0$.
- Function will be minimum if $AC\text{-}B^2 > 0$ *and* $A > 0$ at that stationary point
- Function will be maximum if $AC\text{-}B^2 > 0$ *and* $A < 0$ at that stationary point
- Function will be neither maximum nor minimum if $AC\text{-}B^2 < 0$ at that stationary point and it is called as *SADDLE POINT.*

25. Explain Maxima & Minima for Functions of Two Variables& hence obtain the Necessary Conditions for Maxima, Minima.

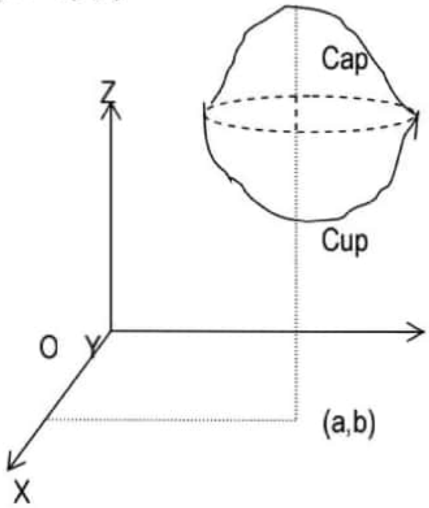
Solution: Let Z = f (x,y) be a given function of two independent variables x & y. The above equation represents a surface in 3D.



A given points (a,b) on the surface has Co-ordinates [a,b, f (a,b)]

Definition:

The function Z = f (x,y) is said to be a maximum at the point (a,b) if f (x,y) < f (a,b) in the neighborhood of the point (a,b)



Definition:

The function Z = f (x,y) is said to posses a minimum at the point (a,b) if f (x,y) > f (a,b) in the neighborhood of the point (a,b)

Necessary Condition for Maxima, Minima:

If Z = f (x,y) has a max or min at (a,b) then $f_x(a,b) = 0$, $f_y(a,b) = 0$

Sufficient Conditions for Maxima, Minima:

Put $R = f_{xx}(a,b)$, $S = f_{xy}(a,b)$, $T = f_{yy}(a,b)$

(1) Suppose $S^2 - RT > 0$

There is no maxima or minima at (a,b)

(2) Suppose $S^2 - RT < 0$

Thus there is maxima or minima at (a,b) according as R < 0 Or R > 0

(3) Suppose $S^2 - RT = 0$, Then there is a saddle point at (a,b)

26. Find the maxima and minima of the functions $f(x,y) = x^3 + y^3 - 3axy$, $a > 0$ is constant.

Solution: Given $f(x,y) = x^3 + y^3 - 3axy$

$f_x = 3x^2 - 3ay$, $f_y = 3y^2 - 3ax$

$f_{xx} = 6x$ $f_{yy} = 6y$.

Put $f_x = 0$, $f_y = 0$ and solve

i.e., $3x^2 - 3ay = 0$ & $3y^2 - 3ax = 0$

i.e., $x^2 = ay$ & $y^2 = ax$

$\Rightarrow y = \frac{x^2}{a}$ $\quad\therefore \Rightarrow \left(\frac{x^2}{a}\right)^2 = ax$ $\quad (\because x^2 = ay)$

$\therefore \frac{x^4}{a^2} = ax$

$\therefore x^4 = a^3x$

i.e., $x(x^3 - a^3) = 0$

$\therefore x = 0, x = a$

$\Rightarrow y = 0, y = \pm a$

$\therefore$ The critical Or stationary points are (0,0), (a,a) and (a,-a)

(1) At (0,0)

$R = f_{xx}(0,0) = 0$

$S = f_{xy}(0,0) = -3a$

$T = f_{yy}(0,0) = 0$

$\therefore S^2 - RT = 9a^2 - 0 = 9a^2 > 0$

$\therefore$ There is neither a maximum or a minimum at (0,0)

27. Examine the following functions for extreme values $f = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

Solution:

$f_x = 4x^3 - 4x + 4y$

$f_y = 4y^3 - 4x - 4y$

$f_{xy} = 4, \quad f_{xx} = 12x^2 - 4, \quad f_{yy} = 12y^2 - 4$

Put $f_x = 0$, $f_y = 0$ and solve

i.e., $4x^3 - 4x + 4y = 0 \qquad \rightarrow (1)$

$4y^3 + 4x - 4y = 0 \qquad \rightarrow (2)$

Adding (1) & (2), we get

$4(x^3 + y^3) = 0$

i.e., $x^3 + y^3 = 0$

i.e., $y = -x$

Substitute $y = -x$ in (1), we get

$4x^3 - 4x - 4x = 0$

i.e., $4x^3 - 8x = 0$

i.e., $x^3 - 2x = 0 \Rightarrow x(x^2 - 2) = 0$

i.e., $x = 0$ & $x^2 - 2 = 0$

i.e., $x = 0$ & $x = \pm\sqrt{2}$

$x = \sqrt{2}, -\sqrt{2}$

$\therefore x = 0, \sqrt{2}, -\sqrt{2}$ and corresponding values of y are $y = 0, -\sqrt{2}, \sqrt{2}$

$\therefore$ The critical points are $(0,0)$, $\left(\sqrt{2}, -\sqrt{2}\right)$, $\left(-\sqrt{2}, \sqrt{2}\right)$

(1) at (0,0)

$R = f_{xx}(0,0) = -4$

$S = f_{xy}(0,0) = 4$

$T = f_{yy}(0,0) = -4$

$\therefore S^2 - RT = 16 - (-4)(-4) = 16 - 16 = 0$

i.e., $S^2 - RT = 0$, These is a saddle point at (0,0)

(2) at $\left(\sqrt{2}, -\sqrt{2}\right)$

$R = f_{xx}\left(\sqrt{2}, -\sqrt{2}\right) = 24 - 4 = 20$

$S = f_{xy}\left(\sqrt{2}, -\sqrt{2}\right) = 4$

$T = f_{yy}\left(\sqrt{2}, -\sqrt{2}\right) = 20$

$\therefore S^2 - RT = 16 - (20)(20) = 16 - 400 = -384 < 0$

Thus these is neither maximum nor minimum according to $R < 0$ or $R > 0$ at $\left(\sqrt{2}, -\sqrt{2}\right)$

Hence $R = 20 > 0$

$\therefore$ There is a minimum at $\left(\sqrt{2}, -\sqrt{2}\right)$

$$\therefore f_{min} = \left(\sqrt{2}\right)^4 + \left(-\sqrt{2}\right)^4 - 2\left(\sqrt{2}\right)^2 + 4\sqrt{2}\left(-\sqrt{2}\right) - 2\left(-\sqrt{2}\right)^2$$

$= 4 + 4 - 4 - 8 - 4$

$= -8$

(3) at $\left(-\sqrt{2}, \sqrt{2}\right)$

$R = f_{xx}\left(-\sqrt{2}, \sqrt{2}\right) = 20 > 0$

$S = f_{xy}\left(-\sqrt{2}, \sqrt{2}\right) = 4$

$T = f_{yy}\left(-\sqrt{2}, \sqrt{2}\right) = 20$

$\therefore$ S2 – RT = 16 – 400 = -384 < 0

Since R > 0, $\therefore$ There is minima at $\left(-\sqrt{2},\sqrt{2}\right)$

$\therefore$ f_{min} = - 8 at $\left(-\sqrt{2},\sqrt{2}\right)$

$\therefore$ Extreme Value = - 8 at $\left(-\sqrt{2},\sqrt{2}\right)$ & $\left(-\sqrt{2},\sqrt{2}\right)$

Exercise:

1) **Find the extreme values of $f = x^3 y^2 (1 - x - y)$**
2) **Determine the maxima or minima of the function Sin x + Sin y + Sin (x + y)**
3) **Examine the function f(x,y) = 1+ sin($x^2 + y^2$) for extremum.**

28. **If PV^2 = K and if the relative errors in P is 0.05 and in V is 0.025 show that the error in K is 10%.**

Solution : $PV^2 = K$ by data. Also $\dfrac{\delta P}{P} = 0.05$ and $\dfrac{\delta V}{V} = 0.025$

$$\Rightarrow \log P + 2\log V = \log K$$

$$\Rightarrow \delta(\log P) + 2\delta(\log V) = \delta(\log K)$$

$$i.e.,\ \frac{1}{p}\delta P + 2\cdot\frac{1}{V}\delta V = \frac{1}{K}\delta K$$

$$i.e.,\ 0.05 + 2(0.025) = \frac{\delta K}{K} \text{ or } \frac{\delta K}{K} = 0.1$$

$$\therefore \frac{\delta K}{K}\times 100 = (0.1)\times 100 = 10$$

Thus the error in **K** is 10%.

29. **The time T of a complete oscillation of a simple pendulum is given by the formula** $T = 2\pi\sqrt{l/g}\cdot$

(i) **If g is a constant find the error in the calculated value of T due to an error of 3% in the value of l.**

(ii) **Find the maximum error in T due to possible errors upto 1% in l and 3% in g.**

Solution :

$$(i)\quad T = 2\pi\sqrt{l/g}, \qquad g = \text{Constant}, \quad \frac{\delta l}{l}\times 100 = 3$$

$$\Rightarrow \log T = \log 2\pi + \frac{1}{2}(\log l - \log\ g)$$

$$\Rightarrow \delta(\log T) = \delta(\log 2\pi) + \frac{1}{2}\delta(\log g)$$

$$i.e., \frac{\delta T}{T} = 0 + \frac{1}{2}\frac{\delta l}{l} - 0$$

$$or\ \frac{\delta T}{T}\times 100 = \frac{1}{2}\left(\frac{\delta l}{l}\times 100\right) = \frac{1}{2}(3) = 1.5$$

∴ **the error in T = 1.5%.**

(ii) If g is not a constant we have,

$$\frac{\delta T}{T}\times 100 = \frac{1}{2}\left(\frac{\delta l}{l}\times 100\right) - \frac{1}{2}\left(\frac{\delta g}{g}\times 100\right)$$

The error in T will be maximum if the error in l is positive and the error in g is negative (or vice-versa) as the difference in errors converts in to a sum.

$$\therefore \max\left(\frac{\delta T}{T}\times 100\right) = \frac{1}{2}(+1) - \frac{1}{2}(-3) = 2$$

∴ **the maximum error in T is 2%.**

30. **The current measured by a tangent galvanometer is given by the relation $c = k \tan\theta$ where θ is the angle of deflection. Show that the relative error in c due to a given error in θ is minimum when $\theta = 45^0$.**

Solution : Consider $c = k \tan\theta$. K is taken as a constant.

$$\Rightarrow \log c = \log k + \log(\tan\theta)$$

$$\Rightarrow \delta(\log c) = \delta(\log k) + \delta \log(\tan\theta)$$

$$i.e., \frac{1}{c}\delta c = 0 + \frac{\sec^2\theta}{\tan\theta}\delta\theta$$

$$i.e., \frac{\delta c}{c} = \frac{\cos\theta}{\sin\theta}\cdot\frac{1}{\cos^2\theta}\delta\theta \quad or \quad \frac{\delta\theta}{c} = \frac{\delta\theta}{\sin\theta\cos\theta}$$

$$i.e., \frac{\delta c}{c} = \frac{2}{\sin 2\theta}\delta\theta$$

The relative error in c being $\delta c / c$ minimum when the denominator of the R.H.S. is maximum and the maximum value of a sine function is1.

$$\therefore \sin 2\theta = 1 \Rightarrow 2\theta = 90^0 \ or \ \theta = 45^0 \cdot$$

Thus the relative error in c is minimum when $\theta = 45^0$

31. *If* $T = \frac{1}{2}mv^2$ **is the kinetic energy, find approximately the change in T as m changes from 49 to 49.5 and v changes from 1600 to 1590. 6 Marks**

Solution : We have by data $T = \frac{1}{2}mv^2$ and

$$m = 49, m + \delta m = 49.5 \quad \therefore \quad \delta m = 0.5$$

$$v = 1600, v + \delta v = 1590 \therefore \delta v = -10$$

We have to find δT. (logarithm is not required)

$$\therefore \ \delta T = \frac{1}{2}\delta\,(mv^2)$$

$$\frac{1}{2}\left\{m(2v\delta v) + \delta m.v^2\right\}$$

$$i.e, = \frac{1}{2}\left\{(49)\ (2)\ (1600)\ (-10) + (0.5)\ (1600)^2\right\} = 1{,}44{,}000$$

Thus the change in $T = \delta T = -1{,}44{,}000$

32. **The pressure p and the volume v of a gas are concentrated by the relation $pv^{1.4} = constant$. Find the percentage increase in pressure corresponding to a diminution of ½% in volume.**

Solution :

$pv^{1.4} = \text{Constant} = c(say)$, by data.

$$\Rightarrow \ \log p + 1.4\ \log\ v = \log c$$

$$\Rightarrow \ \delta(\log p) + 1.4\delta(\log v) = \delta(\log c)$$

$i.e,\ \dfrac{\delta p}{p} + 1.4\left(\dfrac{\delta v}{v}\right) = 0;$ But $\dfrac{\delta v}{v} \times 100 = -\dfrac{1}{2}$, by data.

$$\therefore \frac{\delta p}{p} \times 100 + 1.4\left(\frac{\delta v}{v} \times 100\right) = 0 \ \text{ or } \frac{\delta p}{p} \times 100 = +0.7.$$

Thus the percentage increase in pressure = 0.7

33. **Find the percentage error in the area of an ellipse when an error of +1% is made in measuring the major and minor axis.**

Solution : For the ellipse $x^2/a^2 + y^2/b^2 = 1$ the area (A) is given by π ab where 2a and 2b are the lengths of the major and minor axis.

Let 2a = x and 2b = y.

By data $\dfrac{\delta x}{x} \times 100 = 1, \dfrac{\delta y}{y} \times 100 = 1.$

$$A = \pi ab = \pi \cdot \frac{x}{2} \cdot \frac{y}{2} = \frac{\pi}{4} xy$$

$$\therefore \log A = \log (\pi / 4) + \log x + \log y$$

$$\Rightarrow \delta(\log A) = \delta \log (\pi / 4) + \delta(\log x) + \delta(\log y)$$

$$i.e., \frac{\delta A}{A} = 0 + \frac{\delta x}{x} + \frac{\delta y}{y} \ or \frac{\delta A}{A} \times 100 = \frac{\delta x}{x} \times 100 + \frac{\delta y}{y} \times 100$$

$$\therefore \frac{\delta A}{A} \times 100 = 1 + 1 = 2$$

Thus error in the area = 2%

27. If the sides and angles of a triangle ABC vary in such way that the circum radius remains constant, prove that

$$\frac{\delta a}{\cos A} + \frac{\delta b}{\cos B} + \frac{\delta c}{\cos c} = 0$$

Solution : If the triangle ABC is inscribed in a circle of radius r and if a,b,c respectively denotes the sides opposite to the angles A,B,C we have the sine rule (formula) given by

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2r$$

or $a = 2r \sin A, b = 2r \sin B, c = 2r \sin C$

$$\Rightarrow \delta a = 2r\delta \ (\sin A), \ \delta b = 2r\delta(\sin B), \ \delta c = 2r\delta(\sin C)$$

$$i.e., \ \delta a = 2r \cos \ A\delta A, \ \delta b = 2r \cos B\delta B, \ \delta c = 2r \cos C\delta C$$

or $\frac{\delta a}{\cos A} = 2r\delta A, \frac{\delta b}{\cos B} = 2r\delta B, \frac{\delta c}{\cos C} = 2r\delta C$

Adding all these results we get,

$$\frac{\delta a}{\cos A} + \frac{\delta b}{\cos B} + \frac{\delta c}{\cos C} = 2r(\delta A + \delta B + \delta C) = 2r\delta(A + B + C)$$

But A+B+C = 180 = π radians = constant.

$\Rightarrow \delta(A + B + C) = \delta$ (constant) = 0

Thus $\frac{\delta a}{\cos A} + \frac{\delta b}{\cos B} + \frac{\delta c}{\cos C} = 0$